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ON THE REGIONS OF IMPOSSIBILITY OF MOTIONS IN THE PROBLEM OF THREE BODIES

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SUMMARY

The regions of impossibility of motions are defined for three bodies, alongside with the conditions for their existence and their fundamental properties, for the case, when the modulus of the moment vector of system's quantity motion, which is constant, is not zero.

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1. We shall consider the motion of three bodies P_1, P_2, P_3 with masses m_1, m_2, m_3, attracting one another according to Newton law in the barycentric system of coordinates [1] Oxyz, so oriented that the direction of the axis Oz coincide with the direction of the moment vector of system's quantity of motion, which, as is well known, is a constant quantity. We shall consider the systems, in which the modulus of this vector $C \neq 0$. In such systems the triple collision of bodies according to the Sundman theorem is impossible [2].

The energy integral has the form $T = U + H$, where T is the kinetic energy of the system; $U = f(m_1m_2r_{12}^{-1} + m_1m_3r_{13}^{-1} + m_2m_3r_{23}^{-1})$ is a force function (f is the gravitational constant, r_{ij} is the distance between P_i and P_j); h is the energy constant. From the qualitative point of view, most difficult is (because of the variety of possible motions) the case $h < 0$, for which it is practical to introduce the constant $h' = -h > 0$. From the integral $T = U - h'$ it is easily obtained that the total decomposition of the system is impossible, since

$$\min_{i\neq j} r_{ij} \leqslant \delta = \frac{f}{h'}(m_1m_2 + m_1m_3 + m_2m_3). \qquad (1)$$

The regions of impossibility of motion for each of the three bodies, the conditions of their existence and their main properties are found in the present paper for the case $C \neq 0$, $h < 0$ $(h' > 0)$.

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2. We shall rest on the inequality, linking the moment of inertia I_ζ of a system of n bodies with masses m_i relative to a certain axis $O\zeta$

$$\left(I_\zeta = \sum_{i=1}^{n} m_i(\xi_i^2 + \eta_i^2)\right)$$

in the rectangular system of coordinates $O\xi\eta\zeta$), the kinetic energy T_ζ of system's motion in the projection on a plane perpendicular to the axis $O\zeta$

$$\left(T_\zeta = \frac{1}{2}\sum_{i=1}^{n} m_i(\dot{\xi}_i^2 + \dot{\eta}_i^2)\right),$$

and the moment of system's L_ζ quantity of motion relative to the axis $O\zeta$

$$\left(L_\zeta = \sum_{i=1}^{n} m_i(\xi_i\dot{\eta}_i - \eta_i\dot{\xi}_i)\right).$$

$$I_\zeta T_\zeta \geqslant {}^1/_2 L_\zeta^2. \tag{2}$$

We shall bring forth the scheme of the demonstration of (2). Let us effect the substitution of variables $\xi = \xi'\cos\omega t - \eta'\sin\omega t$, $\eta = \xi'\sin\omega t + \eta'\cos\omega t$, $\zeta = \zeta'$, where ω is an arbitrary but constant angular velocity of system's $O\xi'\eta'\zeta'$ rotation around the axis $O\zeta' = O\zeta$. Let us express I_ζ, L_ζ and T_ζ by new variables. After that it will be found that

$$T_\zeta = {}^1/_2\sum_{i=1}^{n} m_i(\dot{\xi}_i'^2 + \dot{\eta}_i'^2) + \omega L_\zeta - {}^1/_2\omega^2 I_\zeta \geqslant \omega L_\zeta - {}^1/_2\omega^2 I_\zeta.$$

Therefore, $I_\zeta\omega^2 - 2L_\zeta\omega + 2T_\zeta \geqslant 0$ for any ω. At $I_\zeta = 0$, $L_\zeta = 0$ also, and therefore (2) is fulfilled. This is why we may consider that $I_\zeta > 0$. From the inequality for the trinomial of the second degree it may be seen that it cannot have different real roots, i. e., its discriminant is nonpositive, which precisely leads to (2).

Since $T \geqslant T_\zeta$, $I_\zeta T \geqslant {}^1/_2 L_\zeta^2$. Hence, applicably to the problem of three bodies, it follows that

$$I_\zeta(U - h') \geqslant {}^1/_2 C^2 \sin^2\varphi, \tag{3}$$

provided the axis $O\zeta$ constitutes with the plane Oxy the angle ϕ. By virtue of the choice of the system of coordinates, the plane Oxy is called Laplacian [1].

It is easy to derive from (3) that the location of the three bodies on a single straight line and in particular, the double collision, are possible only in the case of this line's belonging to the Laplacian plane.

3. Let us now pass to the fundamental aim of the present paper. We shall denote the numbers 1, 2, 3, taken in arbitrary but fixed order, by indices i, j, k. We shall call the body P_i as remote (and the bodies P_j and P_k as close (short-range)), provided $r_{ij} \geqslant r_{jk}, r_{ik}$ (in particular cases two or each of the three bodies may be remote (long-range)).

In this section we shall find the limitation to the spatial motion of the body P_i in the assumption that it is remote. After that it will not be difficult to consider the cases when P_i is close with either P_j or P_k. The region of impossibility of motion of the body P_i, common for the three cases, will be the unconditional region of impossibility of motion of the body P_i. Since the index i may assume any of the values 1, 2, 3, we shall find these regions for each of these bodies.

Thus, assume that P_i is a remote body. Then $r_{jk} \leqslant \delta$, where δ is given by (1). Let us denote OP_i by r_i, and the angle between OP_i and the plane Oxy by ϕ_i. We shall apply (3) to the axis OP_i, considering that $\phi_i \neq 0$ (at $\phi_i = 0$ we will not obtain the limitation on r_i). The mass center $O_{(jk)}$ of the bodies P_j and P_k is on the straight line OP_i in the opposite direction from the body P_i at a distance from it equal to $\rho_i = mr_i/(m_j + m_k)$, where m is the mass of the whole system. The distances of P_j from $O(jk)$ and of P_k from $O(jk)$ are equal:

$$r_{j(jk)} = m_k r_{jk}/(m_j + m_k) \leqslant m_k\delta/(m_j + m_k), \quad r_{k(jk)} = m_j r_{jk}/(m_j + m_k) \leqslant m_j\delta/(m_j + m_k).$$

It is evident that

$$r_{ij} \geqslant \rho_i - r_{j(jk)} \geqslant (mr_i - m_k\delta)/(m_j + m_k), \quad r_{ik} \geqslant \rho_i - r_{k(jk)} \geqslant (mr_i - m_j\delta)/(m_j + m_k).$$

The right-hand parts of these estimates are positive for $r_i > A_i$, where $A_i = \delta \max (m_j, m_k)/m$. These are the only values of r_i we shall consider (the opposite inequality $r_i \leqslant A_i$ already constitutes a limitation on r_i). For $\phi_i \neq 0$ the bodies cannot lie on the same straight line. Let us draw in the plane $OP_iP_jP_k$ a straight line perpendicular to $O(jk)OP_i$, and passing through the point $O(jk)$. We shall denote by l_{jk} the projection of the segment P_jP_k on this straight line, so that $r_{jk} \geqslant l_{jk} > 0$. The moment of inertia of the system relative to the axis OP_i is $I_i = m_j m_k l_{jk}^2/(m_j + m_k)$. For the force function we obtain the estimate

$$U = f(m_j m_k r_{jk}^{-1} + m_i m_j r_{ij}^{-1} + m_i m_k r_{ik}^{-1}) < f\{m_j m_k l_{jk}^{-1} + m_i(m_j + m_k) \times [m_j/(mr_i - m_k\delta) + m_k/(mr_i - m_j\delta)]\}.$$

As a result, from (3) we shall obtain

$$\frac{m_j m_k}{m_j + m_k}\left[h' - fm_i(m_j + m_k)\left(\frac{m_j}{mr_i - m_k\delta} + \frac{m_k}{mr_i - m_j\delta}\right)\right] l_{jk}^2 - \frac{fm_j^2 m_k^2}{m_j + m_k} l_{jk} + \frac{C^2}{2}\sin^2\varphi_i < 0. \tag{4}$$

Let us denote $F_i(r_i) = m_j/(mr_i - m_k\delta) + m_k/(mr_i - m_j\delta)$. As r_i varies from A_i to $+\infty$, F_i decreases monotonically from $+\infty$ to zero. For this reason there is only a unique value A_i', $A_i < A_i' < +\infty$ such that $F_i(A_i') = h'/fm_i(m_j + m_k)$. We shall consider only $r_i > A_i'$. Then the previous condition $r_i > A_i$ is fulfilled automatically and the coefficient at l_{jk}^2 in (4) is positive. Because of the inequality (4), the trinomial of the second degree has two different real roots, i. e., a positive discriminant. This leads to the inequality $F_i(r_i) > \Phi_i(\sin\phi_i)$ sought for, where $\Phi_i(u) = [h' - f^2 m_j^3 m_k^3/2(m_j + m_k)C^2u^2]/ fm_i(m_j + m_k)$. As $|\phi_i|$ varies from 0 to $\pi/2$, $\Phi_i(\sin\phi_i)$ monotonically increases from $-\infty$ to $\Phi_i(1) < F_i(A_i')$. If $h'C^2 > f^2 m_j^3 m_k^3/2(m_j + m_k)$, $\Phi_i(\sin\phi_i)$ will pass through zero at $|\varphi_i| = \varphi_i^* = \arcsin fm_j m_k[m_j m_k/2(m_j + m_k)h'C^2]^{1/2}$, $0 < \varphi_i^* < \pi/2$.

It is clear that for $0 \leqslant |\phi_i| \leqslant \phi_i^*$ no limitation will be put on r_i. For $\varphi_i^* < |\varphi_i| \leqslant \pi/2$, there will appear, to the contrary, a limitation above by r_i: $r_i < R_i(\phi_i)$ for $r_i > A_i'$, $\varphi_i^* < |\varphi_i| \leqslant \pi/2$. It follows from the properties of $F_i(r_i)$ and $\Phi_i(\sin\phi_i)$ that as $|\phi_i|$ varies from ϕ_i^* to $\pi/2$, $R_i(\phi_i)$ decreases monotonically from $+\infty$ to $R_i(\pi/2)$. Since $F_i(A_i') > \Phi_i(1)$, $R_i(\pi/2) > A_i'$. Consequently, in the result obtained by us

$$r_i < R_i(\varphi_i), \ \varphi_i^* < |\varphi_i| \leqslant \pi/2, \tag{5}$$

the condition $r_i > A_i'$ may be removed (from $r_i \leqslant A_i'$ (5) follows all the more).

It remains to reduce the explicit expression (5). Under the condition $r_i > A_i$ from $F_i(r_i) > \Phi_i(\sin\phi_i)$ follows the inequality:

$$r_i^2 - \frac{m_j + m_k}{m}\left[\delta + \frac{1}{\Phi_i(\sin\varphi_i)}\right] r_i + \frac{\delta}{m^2}\left[m_j m_k \delta + \frac{m_j^2 + m_k^2}{\Phi_i(\sin\varphi_i)}\right] < 0. \tag{6}$$

The trinomial of the second degree (6) has two unequal positive roots: $\tilde{R}_i(\phi_i)$ and $R_i(\phi_i)$, with $\tilde{R}_i < R_i$, The solution of inequality (6) has the form

$$\tilde{R}_i < r_i < R_i.$$

It follows from the properties of $F_i(r_i)$ that $\tilde{R}_i < A_i$ at all times. This is why

$$A_i < r_i < R_i,$$

for, (6) has been derived in the assumption that $r_i > A_i$. But now we may discard it. Thus, (5) is valid, where

$$R_i(\varphi_i) = \frac{m_j + m_k}{2m}\left[\delta + \frac{1}{\Phi_i(\sin\varphi_i)}\right] + \\ + \left\{\frac{(m_j + m_k)^2}{4m^2}\left[\delta + \frac{1}{\Phi_i(\sin\varphi_i)}\right]^2 - \frac{\delta}{m^2}\left[m_j m_k \delta + \frac{m_j^2 + m_k^2}{\Phi_i(\sin\varphi_i)}\right]\right\}^{1/2}. \tag{7}$$

For the factual construction of the boundary of (5) it is more practical, however, to reduce the inequality $F_i(r_i) > \phi_i(\sin\phi_i)$ to the form

$$\sin\varphi_i| < f m_j m_k \left\{\frac{m_j m_k}{2(m_j + m_k)C^2[h' - f m_i(m_j + m_k)F_i(r_i)]}\right\}^{1/2}. \tag{8}$$

4. Assume now that P_i is close to P_j, so that $r_{ij} \leqslant \delta$. The distance of P_i from $O(ij)$ is $r_{i(ij)} = m_j r_{ij}/(m_i + m_j) \leqslant m_j\delta/(m_i + m_j)$. Since P_k is a long-range body $r_k < R_k(\varphi_k)$, and $\varphi_k^* < |\varphi_k| \leqslant \pi/2$. Hence for $O_{(ij)}$: $r_{(ij)} < m_k R_k[\varphi_{(ij)}]/(m_i + m_j)$. $\varphi_k^* < |\varphi_{ij}| \leqslant \pi/2$. Therefore

$$r_i < R_i^j(\varphi_i), \quad \varphi_k^* < |\varphi_i| \leqslant \pi/2, \tag{9}$$

where $R_i^{\,j}(\phi_i)$ passes beyond the boundary for $\mathcal{O}_{(ij)}$ and parallelwise to it at the range $m_j\delta/(m_i + m_j)$. Analogously, if P_i is close to P_k,

$$r_i < R_i^{\,k}(\varphi_i), \quad \varphi_j^* < |\varphi_i| \leqslant \pi/2, \tag{10}$$

where $R_i^{\,k}(\phi_i)$ passes beyond the boundary for $\mathcal{O}_{(ij)}$ and parallelwise to it at the range $m_k\delta/(m_i + m_k)$.

Choosing for a given $|\phi_i|$ the greatest one, we may construct the unconditional boundary of the region of impossibility of body P_i motion from the right-hand parts of (5), (9), (10). It exists for $\phi^* < |\phi_i| < \pi/2$, where ϕ^*

$$\phi^* = \max(\phi_i^*, \phi_j^*, \phi_k^*) = \max(\phi_1^*, \phi_2^*, \phi_3^*).$$

If we number the masses of the bodies by order of decrease: $m_1 > m_2 > m_3$, it will follow from the expressions for the angles ϕ_i^* that

$$\phi^* = \arcsin fm_1m_2[m_1m_2/\,2(m_1 + m_2)h'C^2]^{1/2}.$$

Thus, the unconditional boundary of the region of impossibility of motion for each of the three bodies will exist, provided

$$h'C^2 > f^2m_1^3m_2^3/2(m_1 + m_2), \tag{11}$$

where m_1, m_2 are the two greatest masses. It is easy to construct examples, in which the condition (11) is really fulfilled. For lack of space we shall limit ourself to the simplest example. For the triangular Lagrange solution with equal masses $\phi^* = \arcsin 1/3/2 \cong 13°38'$. Changing little the initial conditions, one may obtain from the plane motion a spatial one, for which the angle ϕ^* will little differ from the value brought out.

**** T H E   E N D ****

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